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Sensitivity analysis of coincident critical loads with respect to minor imperfection

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Abstract

A new formulation is presented for sensitivity analysis of a coincident critical load factor. Only symmetric elastic structures subjected to symmetric loads are considered, and sensitivity coefficients are found for a symmetric design modification which corresponds to a minor imperfection. It is shown that the formulation for sensitivity analysis of multiple linear buckling load factor can be successfully combined with that of nonlinear buckling to develop a formulation for coincident nonlinear buckling load factor. The proposed formulation is verified by analytical examples of simple spring–bar systems. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Sensitivity analysis; Elastic stability; Coincident buckling; Minor imperfection

1. Introduction

Sensitivity of buckling load factor of an elastic structure with respect to an asymmetric imperfection has been extensively investigated and general forms of imperfection sensitivity analysis have been presented for distributed parameter structures (Koiter, 1945) and for finite dimensional structures (Thompson, 1969; Thompson and Hunt, 1973). It has also been pointed out that imperfection sensitivity can increase due to interaction of buckling modes if two or more critical points coincide or are closely spaced (Ho, 1974; Huseyin, 1975; Thompson and Hunt, 1973; Hutchinson and Amazigo, 1967).

Most of all the studies on sensitivity analysis of buckling load factor, however, concern reduction factor of the critical load due to asymmetric imperfection that is classified as *major imperfection*. Evaluation of sensitivity with respect to a major imperfection is very important to estimate maximum load factor of a structure that has an unavoidable manufacturing or construction error.

In the engineering practice, structures to be built often have symmetry properties, and variation of critical load factor with respect to a symmetric design modification is to be evaluated in the process of redesign or design modification. For a symmetric structure subjected to a set of symmetric loads, which is called *symmetric system* for brevity, symmetric design modification is conceived as *minor imperfection*.

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Roorda (1968) derived analytical formulation of sensitivity analysis of bifurcation load factor with respect to a minor imperfection, and showed that the sensitivity coefficient is bounded for such an imperfection. Ohsaki and Uetani (1996) presented three approaches for sensitivity analysis of bifurcation load factors of symmetric systems, one of which is an explicit form of the formulation by Roorda (1968).

In the field of optimum design, on the other hand, extensive research has been made for computing sensitivity coefficients of linear buckling load factor with respect to design variables such as stiffnesses and nodal locations. Such sensitivity coefficients are called *design sensitivity coefficients* which are used for optimum design and simply for redesigning process. Note that imperfection sensitivity and design sensitivity are virtually identical, and same formulations should be developed. Ohsaki and Nakamura (1994) incorporated the method of imperfection sensitivity analysis for finding optimum designs for specified limit point load factor.

It is well known that optimum designs for specified linear buckling load factor often have multiple or repeated buckling load factors that are nondifferentiable; only directional derivatives or subgradients (Mistakidis and Stavroulakis, 1998) can be defined (Masur, 1984; Haug et al., 1986). Several algorithms have been presented for design sensitivity analysis of multiple buckling load factors (Seyranian et al., 1994), and optimum designs have been obtained for plates and shells under constraints on linear buckling load factor. It has been well discussed in the field of stability analysis that optimization for nonlinear buckling results in coincident buckling that may dramatically increase imperfection sensitivity (Huseyin, 1975; Thompson and Lewis, 1972). Ohsaki (2000) demonstrated that optimization does not always increase imperfection sensitivity even if the optimal design has coincident critical points.

In this paper, a new formulation is presented for sensitivity analysis of coincident buckling load factor of symmetric systems with respect to symmetric design modification which is classified as minor imperfection. The validity of the proposed formulation is demonstrated through the examples of two- and three-degree-of-freedom spring-bar systems.

2. Coincident critical points

Consider an elastic structure discretized by using the finite element method. The vector of proportional loads $\mathbf{P}$ is defined by the vector $\mathbf{P}^0$ of loading pattern and the load factor Λ as

$$\mathbf{P} = \Lambda \mathbf{P}^0. \tag{1}$$

The vector of nodal displacements is denoted by $\mathbf{Q} = \{Q_i\}$. The total potential energy is a function of $\mathbf{Q}$ and Λ which is written as $\Pi^{\mathrm{S}}(\mathbf{Q}, \Lambda)$.

Let S_i denote partial differentiation of Π^{S} with respect to Q_i. Stationary condition of Π^{S} with respect to Q_i leads to the following equilibrium equations:

$$S_i = 0, \quad (i = 1, 2, \ldots, f), \tag{2}$$

where f is the number of degree of freedom of displacements. The path of equilibrium state that originates the undeformed initial state is called fundamental equilibrium path.

The second-order partial differential coefficient of Π^{S} with respect to Q_i and Q_j is denoted by S_{ij}. The matrix $\mathbf{S} = [S_{\mathrm{ij}}]$ is called stability matrix or tangent stiffness matrix. The rth eigenvalue $\lambda^r(\Lambda)$ and eigenvector $\Phi^r(\Lambda)$ of $\mathbf{S}$ along the fundamental equilibrium path are defined by

$$S_{ij}\phi_j^r = \lambda^r \phi_i^r \quad (i = 1, 2, \ldots, f) \tag{3}$$

where ϕ_i^r is the ith component of Φ^r, and the summation convention is used *only* for the subscripts; the superscripts are *not* summed. The eigenmode Φ^r is normalized by

$$\phi_j^r \phi_j^r = 1. \tag{4}$$

An equilibrium state that satisfies Eq. (2) is stable if the lowest eigenvalue λ^1 is greater than 0, and is unstable if $\lambda^1 < 0$. The value of Λ corresponding to $\lambda^1 = 0$ is called buckling or critical load factor, and such an equilibrium state is called critical point. In the following, the values corresponding to the critical point is indicated by a superscript $(\ )^{\mathrm{c}}$. The critical points are classified into a limit point and a bifurcation point which are characterized by $\phi_j^{\mathrm{c}1} S_{j\Lambda} \neq 0$ and $\phi_j^{\mathrm{c}1} S_{j\Lambda} = 0$, respectively, where $(\ )_{\Lambda}$ indicates partial differentiation with respect to Λ.

Consider a case where $\mathbf{S}$ has multiple zero eigenvalues, and the remaining eigenvalues are positive. This type of critical point is called coincident critical point. Let m denote the multiplicity of the fundamental zero eigenvalues. In this case, an arbitrary linear combination of $\Phi^{\mathrm{c}\alpha}$ $(\alpha = 1, 2, \dots, m)$ satisfies the following equation:

$$\sum_{\alpha=1}^{m} S_{ij} a_\alpha \phi_j^{\mathrm{c}\alpha} = 0 \quad (i = 1, 2, \dots, f), \tag{5}$$

where a_α $(\alpha = 1, 2, \dots, m)$ are the coefficients and α and β are used in the following for the values corresponding to zero eigenvalues. Eq. (5) is written simply as

$$S_{ij} \psi_j = 0 \quad (i = 1, 2, \dots, f), \tag{6}$$

where

$$\psi_j = \sum_{\alpha=1}^{m} a_\alpha \phi_j^{\mathrm{c}\alpha}. \tag{7}$$

3. Sensitivity of coincident critical load factor with respect to minor imperfection

Consider a structure which has a plane or an axis of symmetry and the applied proportional loads are also symmetric. In this case, the deformation along the fundamental equilibrium path is symmetric, and this type of system is called *symmetric system* for brevity. We consider a case where the structure reaches a bifurcation point; not a limit point.

Let $\mathbf{b}$ denote the vector of design variables that represents the element stiffnesses such as the thickness of a plate element and the cross-sectional area of a truss element. Sensitivity analysis is carried out for a symmetric system considering symmetric design modification which is defined by a mode vector $\mathbf{b}^{\mathrm{s}}$. A set of symmetric systems in the vicinity of the initial system $\mathbf{b}_0$ is defined by using a design parameter ξ as

$$\mathbf{b} = \mathbf{b}_0 + \xi \mathbf{b}^{\mathrm{s}}. \tag{8}$$

Note that a symmetric design change of a symmetric system corresponds to a minor imperfection, and the sensitivity coefficients with respect to a minor imperfection are bounded (Roorda, 1968). The critical point of an imperfect system is also a bifurcation point if a minor imperfection is considered, whereas the critical point may turn out to be a limit point for an asymmetric major imperfection.

Let Q_i^{c} denote the displacements at the critical point. The values Q_i^{c}, Λ^{c}, $\lambda^{\mathrm{c}r}$ and $\Phi^{\mathrm{c}r}$ at the critical point are functions of ξ which are indicated by a hat as $\hat{Q}_i^{\mathrm{c}}(\xi)$, $\hat{\Lambda}^{\mathrm{c}}(\xi)$, $\hat{\lambda}^{\mathrm{c}r}(\xi)$ and $\hat{\Phi}^{\mathrm{c}r}(\xi)$. A hat is also used for other quantities to indicate their dependence on ξ. In the following, all the variables are evaluated at the critical point of the perfect system with $\xi = 0$, e.g., $Q_i^{\mathrm{c}} = \hat{Q}_i^{\mathrm{c}}(0)$, $\phi_j^{\mathrm{c}r} = \hat{\phi}_j^{\mathrm{c}r}(0)$, and the argument ξ is omitted for brevity.

Various formulations may be possible for defining a symmetric system. Masur (1970) defined a *completely symmetric system* by the condition $S_{ijk} \phi_i^{\mathrm{c}r} \phi_j^{\mathrm{c}s} \phi_k^{\mathrm{c}t} = 0$ for all the possible sets of three modes. This is a rather strong condition for symmetry. The terms *semi-symmetry* and *individual symmetry* are also used for characterizing symmetricity of structures (Thompson, 1984; Huseyin, 1975).

In this paper, we consider a weaker condition for a structure with coincident buckling concerning the prebuckling deformation and the modes corresponding to the vanishing eigenvalue. It is assumed here that the prebuckling deformation is orthogonal to the buckling modes; i.e.,

$$Q_i^{\mathrm{c}} \phi_i^{\mathrm{c}\alpha} = 0 \quad (\alpha = 1, 2, \ldots, m). \tag{9}$$

Since the coincident critical point consists of bifurcation points,

$$S_{i\Lambda} \phi_i^{\mathrm{c}\alpha} = 0 \quad (\alpha = 1, 2, \ldots, m) \tag{10}$$

is satisfied. Note that the critical point can be an asymmetric bifurcation point under the condition that Eq. (9) is satisfied.

An explicit differentiation with respect to ξ is denoted by $(\bar{\ })_{\xi}$; i.e., Q_i^{c} and Λ^{c} are fixed during partial differentiation with respect to ξ. Differentiation of Eqs. (2) and (6) as well as Eq. (4) for $r = \alpha$ with respect to ξ leads to

$$S_{ij} \hat{Q}_j^{\mathrm{c}'} + \bar{S}_{i\xi} + S_{i\Lambda} \hat{\Lambda}^{\mathrm{c}'} = 0, \tag{11}$$

$$S_{ijk} \hat{Q}_k^{\mathrm{c}'} \psi_j^{\mathrm{c}} + S_{ij} \hat{\psi}_j^{\mathrm{c}'} + \bar{S}_{ij\xi} \psi_j^{\mathrm{c}} + S_{ij\Lambda} \psi_j^{\mathrm{c}} \hat{\Lambda}^{\mathrm{c}'} = 0, \tag{12}$$

$$\phi_i^{\mathrm{c}\alpha} \hat{\phi}_i^{\mathrm{c}\alpha'} = 0, \tag{13}$$

where a prime indicates total differentiation with respect to ξ. Note again that summation convention is not used for the superscripts.

By multiplying $\phi_i^{\mathrm{c}\alpha}$ to Eq. (11), the following relation is derived for $\alpha = 1, 2, \ldots, m$:

$$\lambda^{\mathrm{c}\alpha} \hat{Q}_i^{\mathrm{c}'} \phi_i^{\mathrm{c}\alpha} + \bar{S}_{i\xi} \phi_i^{\mathrm{c}\alpha} + S_{i\Lambda} \hat{\Lambda}^{\mathrm{c}'} \phi_i^{\mathrm{c}\alpha} = 0, \tag{14}$$

where Eq. (3) has been used in the first term. The first term of Eq. (14) vanishes because $\lambda^{\mathrm{c}\alpha} = 0$. For the case where the second term is not equal to 0 for $\alpha = 1, 2, \ldots, m$, the absolute value of $\hat{\Lambda}^{\mathrm{c}'}$ should be infinity because $S_{i\Lambda} \phi_i^{\mathrm{c}\alpha}$ in the third term is 0 at a bifurcation point. The imperfection such that $\bar{S}_{i\xi} \phi_i^{\mathrm{c}\alpha} = 0$ is called *minor imperfection* (Roorda, 1968). We consider a case where $\bar{S}_{i\xi} \phi_i^{\mathrm{c}\alpha} = 0$ is satisfied for all the buckling modes $\phi_i^{\mathrm{c}\alpha}$ $(\alpha = 1, 2, \ldots, m)$. This type of imperfection is called *completely minor imperfection* in this paper. It may be observed from Eq. (14) that the sensitivity coefficients of a coincident critical load factor are bounded if a completely minor imperfection is considered.

The following relation is also derived for $r = m + 1, m + 2, \ldots, f$ by multiplying $\phi_i^{\mathrm{c}r}$ to Eq. (11).

$$\lambda^{\mathrm{c}r} \hat{Q}_i^{\mathrm{c}'} \phi_i^{\mathrm{c}r} + \bar{S}_{i\xi} \phi_i^{\mathrm{c}r} + S_{i\Lambda} \hat{\Lambda}^{\mathrm{c}'} \phi_i^{\mathrm{c}r} = 0 \tag{15}$$

where Eq. (3) has been used in the first term. The relation

$$S_{ijk} \hat{Q}_k^{\mathrm{c}'} \phi_j^{\mathrm{c}\beta} \psi_i^{\mathrm{c}} + \bar{S}_{ij\xi} \phi_j^{\mathrm{c}\beta} \psi_i^{\mathrm{c}} + S_{ij\Lambda} \phi_j^{\mathrm{c}\beta} \psi_i^{\mathrm{c}} \hat{\Lambda}^{\mathrm{c}'} = 0 \quad (\beta = 1, 2, \ldots, m) \tag{16}$$

is derived by multiplying $\phi_j^{\mathrm{c}\beta}$ to Eq. (12), where $S_{ij} \phi_j^{\mathrm{c}\beta} = 0$ has been used.

The generalized displacement U_j in the direction of $\hat{\Phi}^{\mathrm{c}j}(0)$ is defined as

$$Q_i = \sum_{j=1}^{f} \hat{\phi}_i^{\mathrm{c}j}(0) U_j, \tag{17}$$

where $\hat{\phi}_i^{\mathrm{c}j}(0)$ is the ith component of $\Phi^{\mathrm{c}j}$ of the perfect system which is normalized by Eq. (4) and the orthonormality condition

$$\hat{\phi}_k^{ci}(0)\hat{\phi}_k^{cj}(0) = \delta_{ij} \tag{18}$$

is to be satisfied, where δ_{ij} is the Kronecker's delta. Note that $\hat{\phi}_i^{cj}(0)$ is used only for transformation of the displacements and is fixed during differentiation with respect to ξ.

The relation between the displacements at the critical point is written as

$$Q_i^c = \sum_{j=1}^{f} \hat{\phi}_i^{cj}(0) U_j^c, \tag{19}$$

where U_j^c is a temporary variable which does not appear in the final form. By multiplying $\hat{\phi}_i^{cr}(0)$ to Eq. (19) and by using Eqs. (18) and (19) may be inversely written as

$$U_r^c = \hat{\phi}_i^{cr}(0) Q_i^c. \tag{20}$$

From Eqs. (19) and (20), the relations between the sensitivity coefficients $\hat{Q}_i^{c'}$ and $\hat{U}_i^{c'}$ are written as

$$\hat{Q}_i^{c'} = \sum_{j=1}^{f} \hat{\phi}_i^{cj}(0) \hat{U}_j^{c'}, \tag{21}$$

$$\hat{U}_r^{c'} = \hat{\phi}_i^{cr}(0) \hat{Q}_i^{c'}. \tag{22}$$

Note again that $\hat{\phi}_i^{cr}(0)$ is fixed during differentiation. In the following, $\hat{\phi}_k^{ci}(0)$ is simply written as ϕ_k^{ci} because all the variables are evaluated at the critical point of the perfect system.

By incorporating Eq. (22) into Eq. (15)

$$\lambda^{cr} \hat{U}_r^{c'} + \bar{S}_{i\xi} \phi_i^{cr} + S_{i\Lambda} \hat{\Lambda}^{c'} \phi_i^{cr} = 0 \tag{23}$$

for $r = m+1, m+2, \ldots, f$ are derived. From Eqs. (9) and (20),

$$U_\alpha^c = 0 \quad (\alpha = 1, 2, \ldots, m), \tag{24}$$

$$\hat{U}_\alpha^{c'} = 0 \quad (\alpha = 1, 2, \ldots, m) \tag{25}$$

are satisfied; i.e., the prebuckling deformation does not have components of the buckling modes corresponding to $\xi = 0$ for any symmetric system defined by the parameter ξ. Then the following relation is also derived from Eqs. (16) and (21):

$$\sum_{r=m+1}^{f} S_{ijk} \phi_k^{cr} \hat{U}_r^{c'} \phi_i^{c\beta} \psi_j^c + \bar{S}_{ij\xi} \phi_j^{c\beta} \psi_i^c + S_{ij\Lambda} \phi_j^{c\beta} \psi_i^c \hat{\Lambda}^{c'} = 0 \quad (\beta = 1, 2, \ldots, m). \tag{26}$$

From Eq. (23), $\hat{U}_r^{c'}$ is written as

$$\hat{U}_r^{c'} = -\frac{1}{\lambda^{cr}} (\bar{S}_{i\xi} \phi_i^{cr} + S_{i\Lambda} \hat{\Lambda}^{c'} \phi_i^{cr}) \quad (r = m+1, m+2, \ldots, f). \tag{27}$$

Incorporating Eq. (27) into Eq. (26), the following formulation of sensitivity coefficient is derived:

$$\hat{\Lambda}^{c'} = \frac{1}{\mu} \left(\bar{S}_{ij\xi} \phi_i^{c\beta} \psi_j^c - \sum_{r=m+1}^{f} \frac{S_{ijk} \phi_i^{c\beta} \psi_j^c \phi_k^{cr} \bar{S}_{l\xi} \phi_l^{cr}}{\lambda^{cr}} \right) \quad (\beta = 1, 2, \ldots, m), \tag{28}$$

where μ is independent of ξ and is given as

$$\mu = \sum_{r=m+1}^{f} \frac{S_{ijk} \phi_i^{c\beta} \psi_j^c \phi_k^{cr} S_{l\Lambda} \phi_l^{cr}}{\lambda^{cr}} - S_{ij\Lambda} \phi_i^{c\beta} \psi_j^c \quad (\beta = 1, 2, \ldots, m). \tag{29}$$

The coefficients a_α for ψ_j^{c} should be specified for computing $\hat{\Lambda}^{\mathrm{c}'}$ from Eq. (28). It is shown in the example of a spring-bar system that the value of $\hat{\Lambda}^{\mathrm{c}'}$ calculated from Eq. (28) depends on the choice of a_α. This is similar to the situation for the multiple eigenvalues of the linear buckling problem. It is well known that the multiple linear buckling load factors are nondifferentiable, and only directional derivatives or subgradients (Mistakidis and Stavroulakis, 1998) can be defined. Accurate directional derivatives should be found if sensitivity analysis is carried out for optimizing structures. Seyranian et al. (1994)) showed that the proper choice of a_α for computing the directional derivatives can be found by solving an eigenvalue problem. Similar formulation is presented as follows for the nonlinear coincident bifurcation load factors.

By using Eqs. (7) and (28) is rewritten as

$$G_{\alpha\beta}a_\alpha = \hat{\Lambda}^{\mathrm{c}'} H_{\alpha\beta}a_\alpha \quad (\beta = 1, 2, \ldots, m), \tag{30}$$

where

$$G_{\alpha\beta} = \bar{S}_{ij\xi}\phi^{\mathrm{c}}_{\beta i}\phi^{\mathrm{c}}_{\alpha j} - \frac{\sum_{r=m+1}^{f} S_{ijk}\phi^{\mathrm{c}}_{\beta i}\phi^{\mathrm{c}}_{\alpha j}\phi^{cr}_{k}\bar{S}_{l\xi}\phi^{cr}_{l}}{\lambda^{cr}}, \tag{31}$$

$$H_{\alpha\beta} = \frac{\sum_{r=m+1}^{f} S_{ijk}\phi^{\mathrm{c}}_{\beta i}\phi^{\mathrm{c}}_{\alpha j}\phi^{cr}_{k}S_{l\Lambda}\phi^{cr}_{l}}{\lambda^{cr}} - S_{ij\Lambda}\phi^{\mathrm{c}}_{\beta i}\phi^{\mathrm{c}}_{\alpha j}. \tag{32}$$

Note that Eq. (30) has an obvious solution $a_\alpha = 0$. In order to find a solution a_α that does not form a null vector, the sensitivity coefficients $\hat{\Lambda}^{\mathrm{c}'}$ and a_α are calculated as a pair of eigenvalue, η and eigenvector, ψ_β, defined by

$$G_{\alpha\beta}\psi_\beta = \eta H_{\alpha\beta}a_\beta \quad (\alpha = 1, 2, \ldots, m), \tag{33}$$

where symmetricity of $G_{\alpha\beta}$ and $H_{\alpha\beta}$ has been used. Relation between directional derivatives and subgradients is investigated in the following example of a two-degree-of-freedom system.

4. Illustrative examples

A two-degree-of-freedom spring-bar system is first considered for concisely illustrating the proposed procedure. However, general applicability of the proposed formulations cannot be proved by this system, because $m = f$ is satisfied. Therefore a three-degree-of-freedom system is next used for the verification for $m < f$.

4.1. A two-degree-of-freedom spring-bar system

Consider a two-degree-of-freedom system as shown in Fig. 1 (Chilver, 1967; Huseyin, 1975) which is supported in the y-direction at nodes 1 and 4, and in x-direction at the center. Note that this system is symmetric with respect to the y-axis. The generalized coordinates q_1 and q_2 are given so that the extensions e_{a}, e_{b}, e_{c} of the springs a, b, c are defined as

$$e_{\mathrm{a}} = \tfrac{1}{2}L(q_1 + q_2), \tag{34}$$

$$e_{\mathrm{b}} = \tfrac{1}{2}L(q_1 - q_2), \tag{35}$$

$$e_{\mathrm{c}} = Lq_1 \tag{36}$$

and the rotation of the spring d is given as

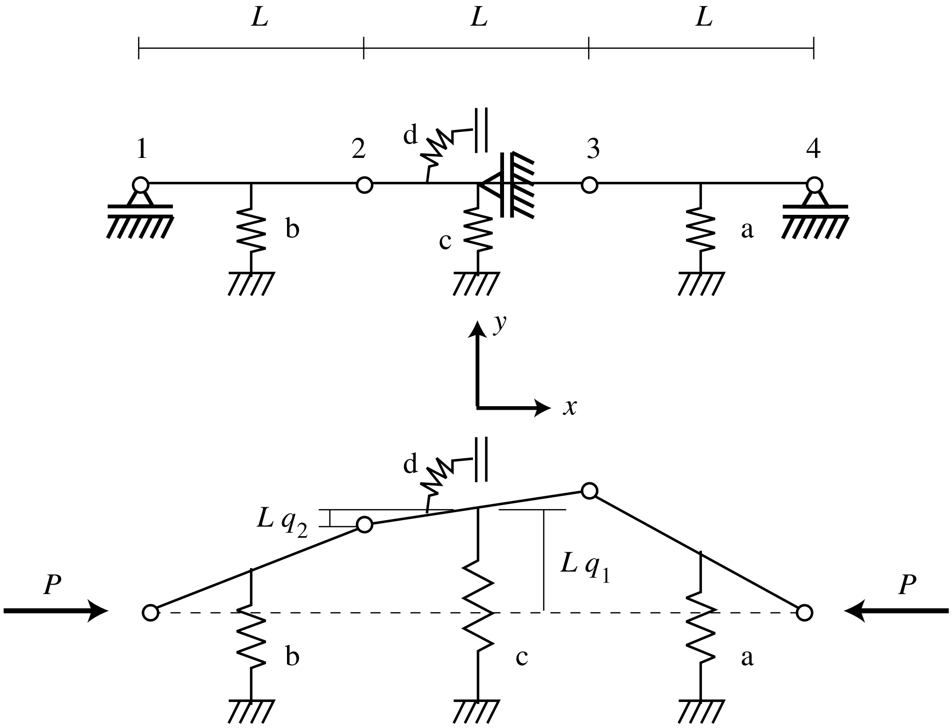

Fig. 1. A two-degree-of-freedom spring-bar system.

$$\theta_{\mathrm{d}} = \sin^{-1} 2q_2. \tag{37}$$

The strain energy is assumed to be defined as

$$E_{\mathrm{a}} = \tfrac{1}{2}A_1(q_1+q_2)^2 + \tfrac{1}{6}B_1(q_1+q_2)^3 + \tfrac{1}{24}C_1(q_1+q_2)^4, \tag{38}$$

$$E_{\mathrm{b}} = \tfrac{1}{2}A_1(q_1-q_2)^2 + \tfrac{1}{6}B_1(q_1-q_2)^3 + \tfrac{1}{24}C_1(q_1-q_2)^4, \tag{39}$$

$$E_{\mathrm{c}} = \tfrac{1}{2}A_2q_1^2 + \tfrac{1}{6}B_2q_1^3 + \tfrac{1}{24}C_2q_1^4, \tag{40}$$

$$E_{\mathrm{d}} = \tfrac{1}{2}\bar{A}_3(\sin^{-1} 2q_2)^2 \simeq \tfrac{1}{2}\bar{A}_3\left(4q_2^2 + \tfrac{16}{3}q_2^4\right) \equiv \tfrac{1}{2}A_3q_2^2 + \tfrac{1}{4!}C_3q_2^4, \tag{41}$$

where A_1, A_2, B_1, etc. are constants. The relative displacement v in the x-direction between nodes 1 and 4 is written as

$$v = L\left[q_1^2 + 3q_2^2 + \tfrac{1}{4}(q_1^4 + 6q_1^2q_2^2 + 9q_2^4)\right]. \tag{42}$$

Finally, the total potential energy $V = E_{\mathrm{a}} + E_{\mathrm{b}} + E_{\mathrm{c}} + E_{\mathrm{d}} - \Lambda Pv$ is a polynomial function of q_1 and q_2 as

$$V = \tfrac{1}{2}(2A_1 + A_2)q_1^2 + \tfrac{1}{2}(2A_1 + A_3)q_2^2 + \tfrac{1}{3!}[(2B_1 + B_2)q_1^3 + 6B_1q_1q_2^2] + \tfrac{1}{4!}[(2C_1 + C_2)q_1^4 + 12C_1q_1^2q_2^2 + (2C_1 + C_3)q_2^4] - \Lambda PL\left[(q_1^2 + 3q_2^2) + \tfrac{1}{4}(q_1^4 + 6q_1^2q_2^2 + q_2^4)\right]. \tag{43}$$

By successively differentiating V with respect to q_1 and q_2, and by substituting the obvious solution $q_1 = q_2 = 0$ of the fundamental equilibrium path, the following relations are derived:

$$S_{11} = 2A_1 + A_2 - 2\Lambda PL, \tag{44}$$

$$S_{12} = S_{21} = 0, \tag{45}$$

$$S_{22} = 2A_1 + A_3 - 6\Lambda PL, \tag{46}$$

$$S_{111} = 2B_1 + B_2, \tag{47}$$

$$S_{112} = 0, \tag{48}$$

$$S_{221} = 2B_1, \tag{49}$$

$$S_{222} = 0. \tag{50}$$

Note that this system is not completely symmetric. From Eqs. (44) and (46), the two critical load factors are

$$\Lambda^{c1} = \frac{2A_1 + A_2}{2PL}, \tag{51}$$

$$\Lambda^{c2} = \frac{2A_1 + A_3}{6PL}, \tag{52}$$

which have the same value if

$$4A_1 + 3A_2 - A_3 = 0. \tag{53}$$

The parameter A_1 is chosen as imperfection parameter or design variable; i.e., $\xi = A_1$. From (51) and (52), the sensitivity coefficients of Λ^{c1} and Λ^{c2} are written as

$$\Lambda^{c1'} = \frac{1}{PL}, \qquad \Lambda^{c2'} = \frac{1}{3PL}. \tag{54}$$

Suppose that $\Lambda^{c1} = \Lambda^{c2}$ is satisfied. The two eigenvectors may be written as

$$\{\phi_1^c\} = \begin{Bmatrix} p_{11} \\ p_{12} \end{Bmatrix}, \qquad \{\phi_2^c\} = \begin{Bmatrix} p_{21} \\ p_{22} \end{Bmatrix}, \tag{55}$$

where Eq. (4) and $p_{11}p_{22} - p_{12}p_{21} \neq 0$ are to be satisfied. Then Eq. (30) is written as

$$\begin{bmatrix} 2 & 2(p_{11}p_{21} + p_{12}p_{22}) \\ 2(p_{11}p_{21} + p_{12}p_{22}) & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \hat{\Lambda}^{c'} \begin{bmatrix} 2PLp_{11}^2 + 6PLp_{12}^2 & 2PLp_{11}p_{21} + 6PLp_{12}p_{22} \\ 2PLp_{11}p_{21} + 6PLp_{12}p_{22} & 2PLp_{21}^2 + 6PLp_{22}^2 \end{bmatrix} \times \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}. \tag{56}$$

It has been confirmed that the same results as Eq. (54) are obtained by carrying out eigenvalue analysis of Eq. (56). Note that the package Maple V Release 5 has been used for symbolic computation, and the details are omitted because use of large space for algebraic expressions should be avoided. For example, if we chose $(p_{11}, p_{12}) = (1, 0)$ and $(p_{21}, p_{22}) = (0, 1)$, Eq. (30) is reduced to

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \hat{\Lambda}^{c'} \begin{bmatrix} 2PL & 0 \\ 0 & 6PL \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \tag{57}$$

and Eq. (54) is immediately derived. In the example above, however, the terms of the summation in Eq. (28) vanish because $m = f = 2$.

Fig. 2 illustrates the relation between the design parameter and the two critical load factors. It may be observed from Fig. 2 that the sensitivity coefficient of Λ^{c1} that is plotted by solid lines is discontinuous at

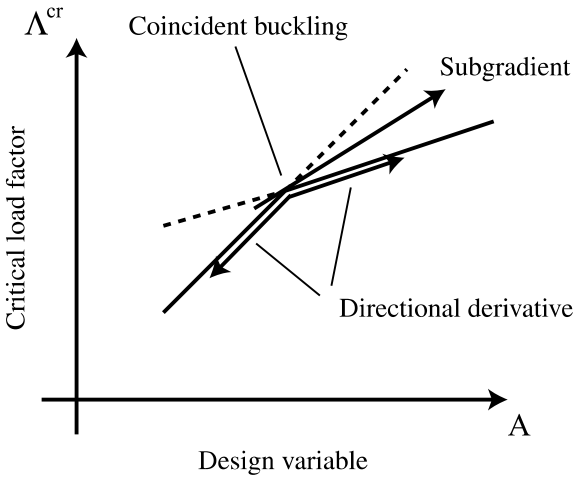

Fig. 2. Definition of directional derivative and subgradient.

the coincident critical point, and the directional derivatives may be defined as illustrated in Fig. 2. Note that directional derivatives should be calculated to accurately estimate the variation of the lowest critical load factor.

For the case of $m = 2$, the two eigenvalues are defined as the maximum and minimum values of the Rayleigh's quotient

$$R = \frac{G_{\alpha\beta}a_\alpha a_\beta}{H_{\alpha\beta}a_\alpha a_\beta} \tag{58}$$

which is derived from Eq. (30). The eigenvector defined by the linear combination of $(p_{11}, p_{12}) = (1, 0)$ and $(p_{21}, p_{22}) = (0, 1)$ is written by using a parameter t as $\{\psi_i^c\} = \{\sin t, \cos t\}^T$. Fig. 3 shows the relation between t and R. It is observed from Fig. 3 that R is a smooth cyclic function of t, and the value of R for each value of t corresponds to a subgradient as illustrated in Fig. 2.

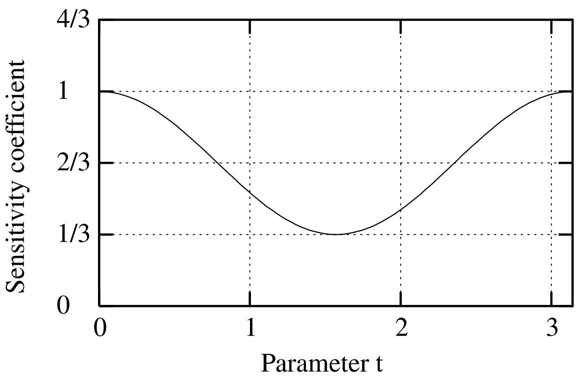

Fig. 3. Relation between mode parameter and nondimensional value *RPL* of Rayleigh's quotient.

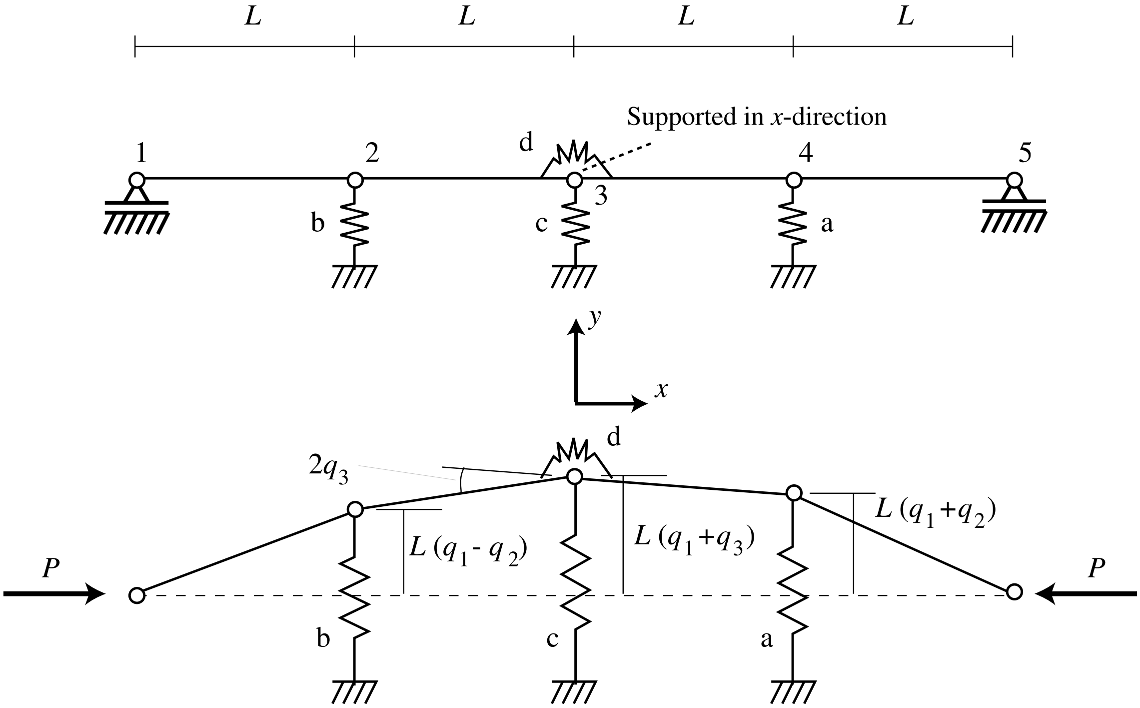

Fig. 4. A three-degree-of-freedom spring-bar system.

4.2. A three-degree-of-freedom spring-bar system

Consider next a three-degree-of-freedom system as shown in Fig. 4 which is supported in the y-direction at nodes 1 and 4, and in x-direction at the center. Note that the support at the center similar to that in Fig. 1 has been omitted for brevity in Fig. 4, and this system is symmetric with respect to the y-axis. The extensions of the springs a, b, c are defined as

$$e_{\mathrm{a}} = L(q_1 + q_2), \tag{59}$$

$$e_{\mathrm{b}} = L(q_1 - q_2), \tag{60}$$

$$e_{\mathrm{c}} = L(q_1 + q_3) \tag{61}$$

and the change of the angle of the spring d is given as

$$\theta_{\mathrm{d}} = 2q_3, \tag{62}$$

where the generalized displacements q_1, q_2, q_3 are the coefficients for the three modes defined as shown in Fig. 5.

The strain energy of each spring is given as

$$E_{\mathrm{a}} = \tfrac{1}{2}A_1 e_{\mathrm{a}}^2 + \tfrac{1}{6}B_1 e_{\mathrm{a}}^3 + \tfrac{1}{24}C_1 e_{\mathrm{a}}^4, \tag{63}$$

$$E_{\mathrm{b}} = \tfrac{1}{2}A_1 e_{\mathrm{b}}^2 + \tfrac{1}{6}B_1 e_{\mathrm{b}}^3 + \tfrac{1}{24}C_1 e_{\mathrm{b}}^4, \tag{64}$$

$$E_{\mathrm{b}} = \tfrac{1}{2}A_2 e_{\mathrm{b}}^2 + \tfrac{1}{6}B_2 e_{\mathrm{b}}^3 + \tfrac{1}{24}C_2 e_{\mathrm{b}}^4, \tag{65}$$

$$E_{\mathrm{d}} = \tfrac{1}{2}D\theta_{\mathrm{d}}^2, \tag{66}$$

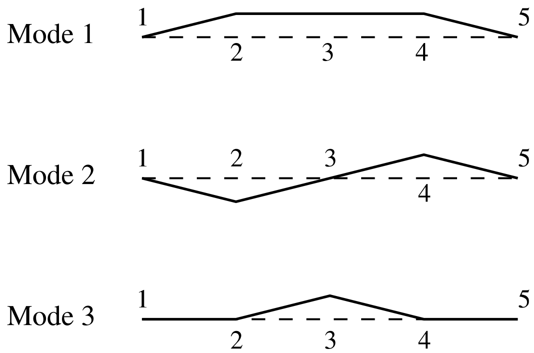

Fig. 5. Displacement mode of the three-degree-of-freedom spring-bar system.

where A_1, D, etc. are constants. The relative displacement v in the x-direction between nodes 1 and 5 is given as

$$v = L\left[q_1^2 + 2q_2^2 + q_3^2 + \tfrac{1}{4}(q_1^4 + 2q_2^4 + q_3^4)\right]. \tag{67}$$

By successively differentiating the total potential energy $V = E_\mathrm{a} + E_\mathrm{b} + E_\mathrm{c} + E_\mathrm{d} - \Lambda P v$ with respect to q_1, q_2 and q_3, and by substituting the obvious solution $q_1 = q_2 = q_3 = 0$ of the fundamental equilibrium path, the following relations are derived:

$$S_{11} = 2A_1 + A_1 - 2\Lambda PL, \tag{68}$$

$$S_{13} = S_{31} = A_1, \tag{69}$$

$$S_{22} = 2A_1 - 2\Lambda PL, \tag{70}$$

$$S_{32} = A_1 + 4D - 2\Lambda PL, \tag{71}$$

$$S_{12} = S_{21} = S_{23} = S_{32} = S_{33} = 0. \tag{72}$$

The third differential coefficients are

$$S_{111} = 3B_1, \tag{73}$$

$$S_{221} = 2B_1, \tag{74}$$

$$S_{311} = S_{331} = S_{333} = B_1, \tag{75}$$

where symmetry in the differentiation such as $S_{221} = S_{212} = S_{122}$ is to be used, and the remaining components are equal to 0. Note that this system is not semi-symmetric.

By carrying out eigenvalue analysis for S_{ij}, the critical load factors are found as

$$\Lambda^{\mathrm{c}1} = \frac{A_1}{2PL}, \tag{76}$$

$$\Lambda^{\mathrm{c}2} = \frac{1}{2PL}[2A_1 + 2D - c], \tag{77}$$

where

$$c = \sqrt{2(A_1)^2 - 4A_1 D + 4D^2}. \tag{78}$$

Therefore two load factors coincide if

$$A_1 = 8D. \tag{79}$$

The parameter A_1 is chosen as design variable; i.e., $\xi = A_1$. From (76a,b), the sensitivity coefficients of Λ^{c1} and Λ^{c2} are written as

$$\Lambda^{c1'} = \frac{1}{2PL}, \tag{80}$$

$$\Lambda^{c2'} = \frac{1}{2PL}\left[1 - \frac{1}{2c}(2A_1 - 4D)\right]. \tag{81}$$

The package Maple V has also been used for symbolic computation. The two lowest eigenvectors of $[S_{ij}]$ are given as

$$\{\phi_1^c\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \qquad \{\phi_2^c\} = \begin{Bmatrix} d/e \\ 0 \\ 1/e \end{Bmatrix}, \tag{82}$$

where

$$d = \frac{-A_1 + 2D - c}{A_1}, \tag{83}$$

$$e = \sqrt{d^2 + 1}. \tag{84}$$

Then Eq. (56) is reduced to

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 - \frac{1}{2c}(2A_1 - 4D) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \hat{\Lambda}^{c'} \begin{bmatrix} 4PL & 0 \\ 0 & 2PL \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \tag{85}$$

and the same results as Eqs. (80) and (81) are derived.

5. Conclusions

A new formulation has been presented for sensitivity analysis of coincident buckling load factor of symmetric systems with respect to symmetric design modification. A weak condition is introduced for defining a symmetric system. In this definition, the prebuckling deformation should be orthogonal to all the buckling modes corresponding to the coincident critical point.

A new concept called *completely minor imperfection* has also been introduced for symmetric systems with coincident bifurcation points. For a completely minor imperfection, the condition for the minor imperfection is to be satisfied by all the coincident buckling modes, and the sensitivity coefficients of a coincident critical load factor are bounded if a completely minor imperfection is considered. A symmetric design modification corresponds to a completely minor imperfection if the system is symmetric.

The formulation for sensitivity analysis of a simple bifurcation load factor with respect to minor imperfection has been extended to the case of coincident buckling. It has been shown that sensitivity analysis of coincident bifurcation load factor can be carried out in a similar manner as multiple linear buckling load factor. The directional derivatives are computed by solving an eigenvalue problem for obtaining the proper choice of the coefficients for the eigenmodes.

Detailed formulations have been presented for a two-degree-of-freedom spring-bar system to illustrate the validity of the proposed formulations. It has been shown that subgradients may be found by incorporating arbitrary set of the coefficients into the Rayleigh's quotient. Since this is a special case where all the buckling load factors coincide, applicability for a general case has been demonstrated by a three-degree-of-freedom spring-bar system.

Application of the proposed method to finite dimensional models with moderately large degree of freedom is to be in a similar manner as shown in the examples in Ohsaki and Uetani (1996). If the undeformed state is taken as the reference configuration for defining the strains, the third order differential coefficients of the total potential energy with respect to the displacements can be easily obtained by using appropriate package of symbolic computation, if necessary.

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